Revision of Linear Algebra and Probability

Advanced Statistical Inference

Simone Rossi

Linear Algebra

1. Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

2. Find the eigenvalues and eigenvectors of the matrix:

$$B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

3. Prove that the set of vectors $\{(1, 2), (2, 4)\}$ is linearly dependent. (Hint: think about the determinant of the matrix formed by these vectors.)

4. Given the matrix $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, compute its inverse, if it exists.

5. Show that the trace of a square matrix is equal to the sum of its eigenvalues. (Hint: use the spectral decomposition and a property of the trace function.)

6. Given the following system of equations, represent it in matrix form:

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x - y + 2z &= 6 \\ -x + 2y + 3z &= 4 \end{aligned}$$

7. Solve the system of equations from the previous question using Cholesky decomposition and the forward/backward substitution method.

Probability

1. Let X be a continuous random variable with probability density function (pdf):

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Is $f(x)$ a valid pdf? Justify your answer.

2. Show that the sum of expectations is equal to the expectation of the sum, i.e., for random variables X and Y, prove that:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

3. Given the probability density function of a normal distribution:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   write down the expression for $\log p(x)$ and simplify it as much as possible.

4. Using Bayes' theorem, derive the posterior distribution $p(y \mid x)$ given the likelihood $p(x \mid y)$ and the prior $p(y)$.

5. Given the probability density function of multivariate normal distribution:

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}\det(\boldsymbol{\Sigma})^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

   write down the expression for $\log p(\boldsymbol{x})$ and simplify it as much as possible.

6. From the previous question, write down a numerically stable expression for $\log p(\boldsymbol{x})$ by avoiding the computation of the determinant and the inverse of the covariance matrix $\boldsymbol{\Sigma}$ directly. (Hint: use the Cholesky decomposition of $\boldsymbol{\Sigma}$.)

7. Given 3 random variables X, Y, and Z, write down all possible expressions to compute the joint probability $p(X, Y, Z)$ using the chain rule of probability.

8. Assume the $\boldsymbol{x}$ is distributed according to a multivariate normal distribution with mean $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}$. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ be a linear transformation matrix. Find the distribution of the transformed variable $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.

9. Let $\boldsymbol{x}$ be a random vector with mean $\boldsymbol{\mu} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^\top$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 6 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

- Compute the marginal distribution $p(x_1, x_3)$.
- Compute the conditional distribution $p(x_1 \mid x_2 = 2, x_3 = 4)$.
- What can you say about the marginal distribution $p(x_2, x_3)$? Are x_2 and x_3 correlated?